Tight Bounds for Algebraic Gossip on Graphs

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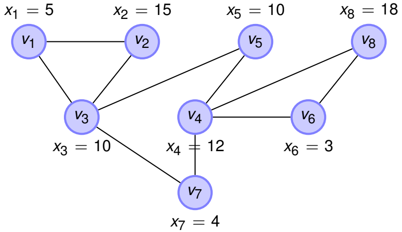
The Task: Disseminate All Values to All Nodes



- Every node needs to learn all the values.
- A node knows only its neighbors.
- A node does not know what values other nodes know.
- We are looking for a local and distributed algorithm.

Time Model

- Time is discrete and divided into timeslots.

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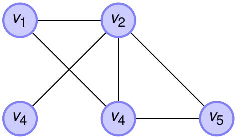

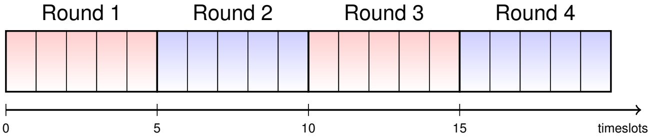

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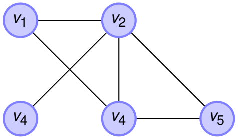
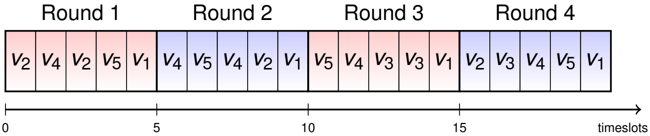
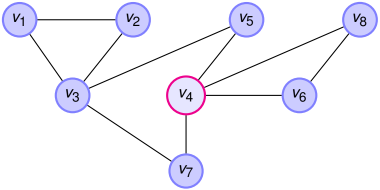




- Every timeslot - **random** node wakes up for a **gossip action**.

Gossip Algorithm – The Way Information is Spread

At each timeslot a **single random node** takes a **gossip action**



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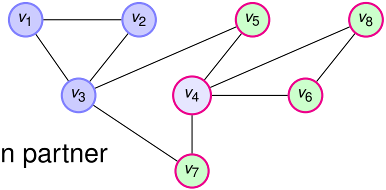

- Gossip Algorithm:
  1. Determines a communication partner randomly among neighbors.
     - Uniform gossip.
     - Non uniform gossip.

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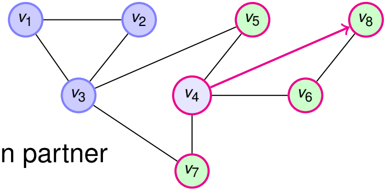
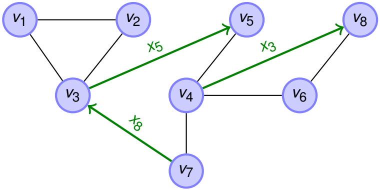


- Gossip Algorithm:
  1. Determines a communication partner randomly among neighbors.
     - Uniform gossip.
     - Non uniform gossip.
  2. Determines how the message is sent.
     - **PUSH** – a message is sent to the partner.
     - **PULL** – a message is sent from the partner.
     - **EXCHANGE** - PUSH and PULL.

Algebraic Gossip is Based on Random Linear Network Coding

instead of sending randomly chosen values...



every message – a single value

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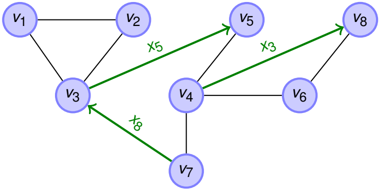
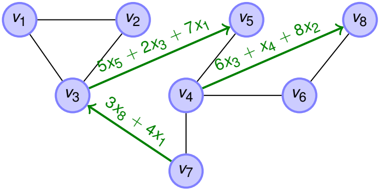


every message – a single value

we send random linear combinations



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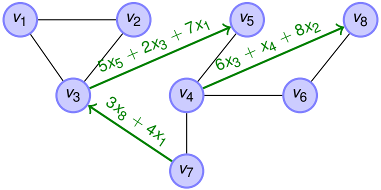

every message – linear equation

linear equations are stored in a matrix form:

$$\begin{bmatrix} 4 & 3 & 7 & 6 & 3 & 0 & 3 & 1 \\ 2 & 0 & 0 & 7 & 4 & 0 & 5 & 0 \\ 1 & 1 & 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 2 & 1 & 5 & 4 & 0 & 0 & 0 \\ 4 & 0 & 7 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 22 \\ 45 \\ 78 \\ 30 \\ 15 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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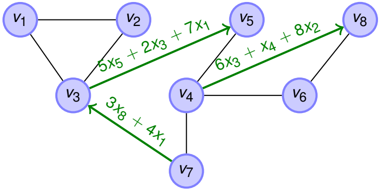

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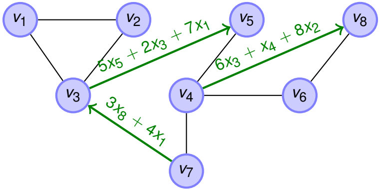

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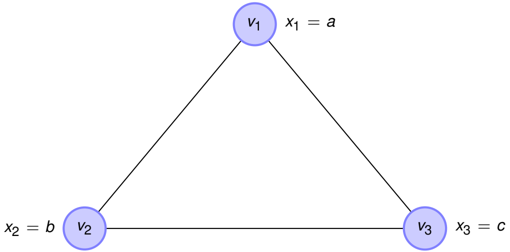
once a node has rank n – it finishes

only **helpful** messages are stored

messages that increase the rank of the matrix

Algebraic Gossip – Simple Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$



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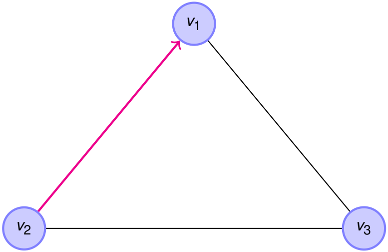

random coefficients

$$\begin{matrix} 3 \\ 2 \\ 5 \end{matrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

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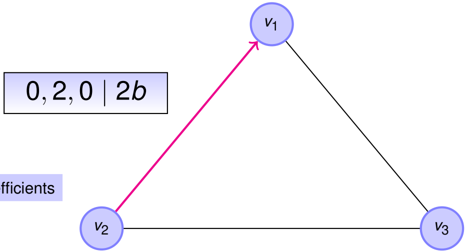

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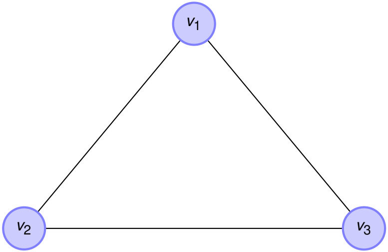

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$$\begin{matrix} 5 \\ 6 \\ 2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 2b \\ 0 \end{bmatrix}$$

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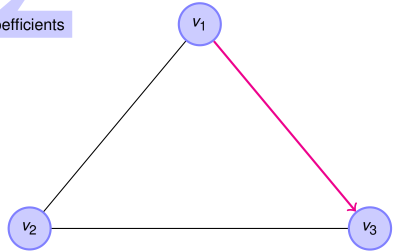

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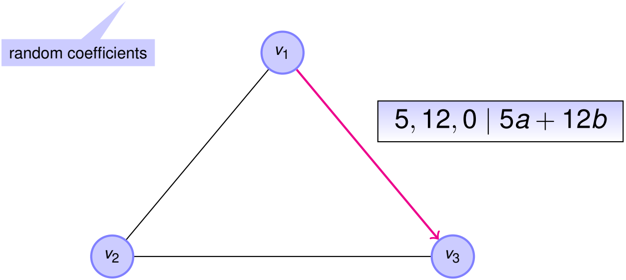

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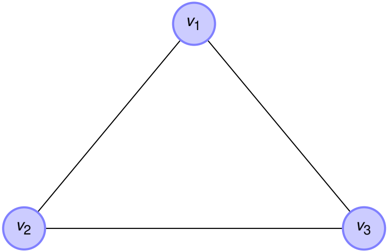

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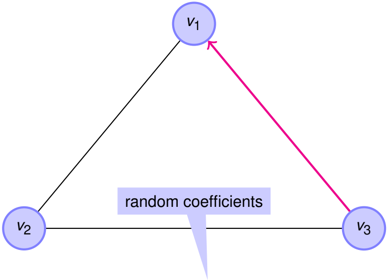

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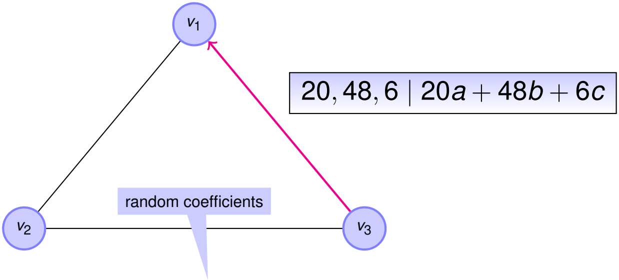

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Algebraic Gossip – Simple Example

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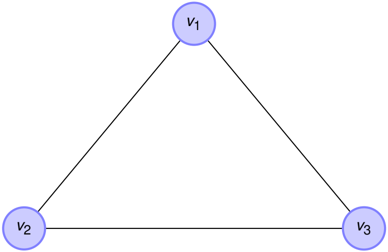

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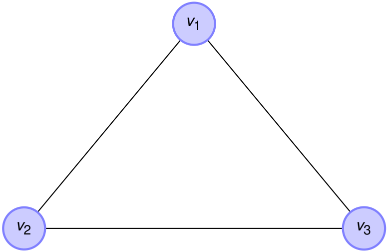
$$\Rightarrow \begin{aligned} x_1 &= a \\ x_2 &= b \\ x_3 &= c \end{aligned}$$



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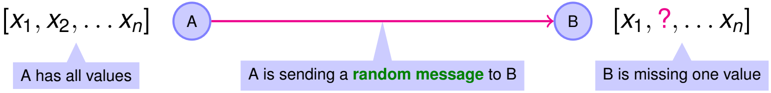
So, Why Algebraic Gossip is Faster?

Without Algebraic Gossip (Random Message Selection)



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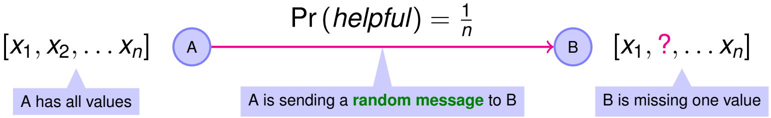

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Without Algebraic Gossip (Random Message Selection)

$[x_1, x_2, \ldots x_n]$ A → B $[x_1, ?, \ldots x_n]$

$\Pr(helpful) = \frac{1}{n}$

A has all values

A is sending a **random message** to B

B is missing one value

With Algebraic Gossip (Random Linear Equations)

[rank n] A → B [rank $n-1$]

A has n independent equations

A is sending a **random linear equation** to B

B has $n-1$ independent equations

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With Algebraic Gossip (Random Linear Equations)

[rank n] A → B [rank $n - 1$]

$\Pr(helpful) = 1 - \frac{1}{q}$

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So, Why Algebraic Gossip is Faster?

Without Algebraic Gossip (Random Message Selection)

$$[x_1, x_2, \ldots x_n] \quad A \xrightarrow{\Pr(helpful) = \frac{1}{n}} B \quad [x_1, ?, \ldots x_n]$$

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With Algebraic Gossip (Random Linear Equations)

$$[\text{rank } n] \quad A \xrightarrow{\Pr(helpful) = 1 - \frac{1}{q}} B \quad [\text{rank } n-1]$$

- A has n independent equations
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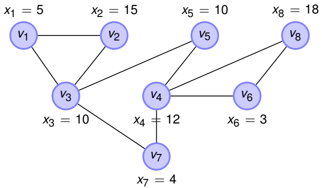
- B has $n-1$ independent equations

$$\Pr(helpful) = \frac{q^n - q^{n-1}}{q^n} = 1 - \frac{1}{q}$$

Research Goal – Stopping Time of Algebraic Gossip



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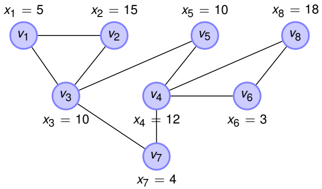

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  - [Deb et al., 2006] – Tight bound for complete graph.

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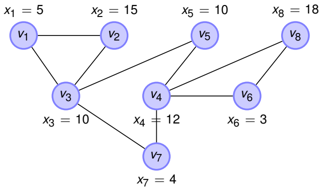

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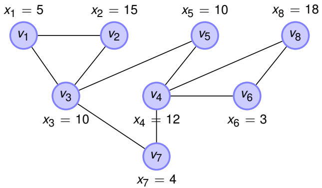

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  - Worst case lower bound was not addressed before.

Our Main Result – Tight Bound for Algebraic Gossip

Theorem 1: Upper Bound

For any graph with n nodes and maximum degree Δ, stopping time of algebraic gossip is $O(\Delta n)$ in expectation and with high probability.

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Corollary 1

For any graph with n nodes,
stopping time of algebraic gossip is $O(n^2)$.

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Corollary 1

For any graph with n nodes,
stopping time of algebraic gossip is $O(n^2)$.

Corollary 2

For any graph with n nodes and
a **constant maximum degree** Δ
stopping time of algebraic gossip is $\Theta(n)$.

Our Main Result – Tight Bound for Algebraic Gossip

Theorem 1: Upper Bound

For any graph with n nodes and maximum degree Δ, stopping time of algebraic gossip is $O(\Delta n)$ in expectation and with high probability.

Theorem 2: Worst Case Lower Bound

For any $2 \leq \Delta \leq \frac{n}{2} + 1$, there exists a graph with maximum degree Δ, for which algebraic gossip takes $\Omega(\Delta n)$ rounds.

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Proof Overview – Very Important Lemma

$$x_1 + x_2 + x_3 + x_4 = 15 \qquad x_1 + x_2 = 10$$
$$2x_3 + 2x_4 = 10 \qquad x_3 + x_4 = 5$$

$$x_1 + x_5 = 8 \qquad x_1 + x_2 = 10$$
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Definition

A node v is called **helpful** to a node u,
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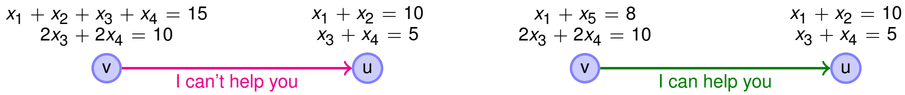
Proof Overview – Very Important Lemma



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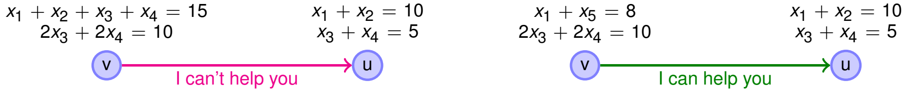

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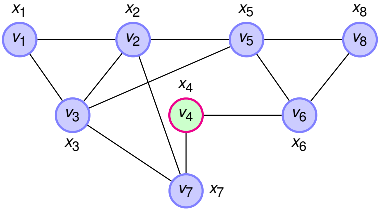
q is a field size from which random coefficients are drawn

Proof Overview – Converting a Graph to a System of Queues

initial graph with an arbitrary node v_4



when v_4 will finish the protocol?

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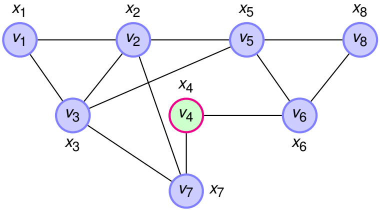

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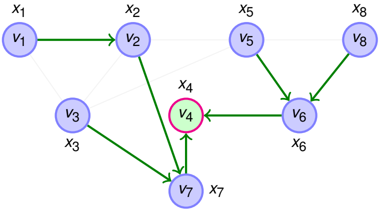
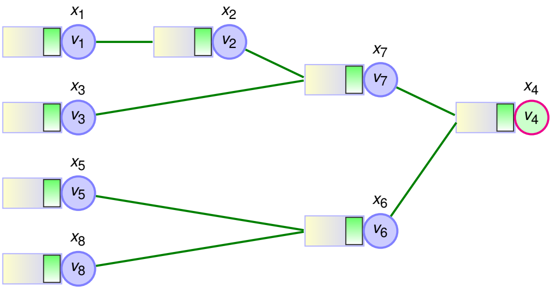
BFS spanning tree rooted at v_4



we **ignore** the messages coming from other edges

Proof Overview – Converting a Graph to a System of Queues



customers are **helpful** messages

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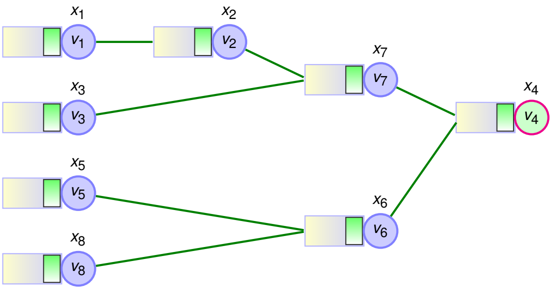

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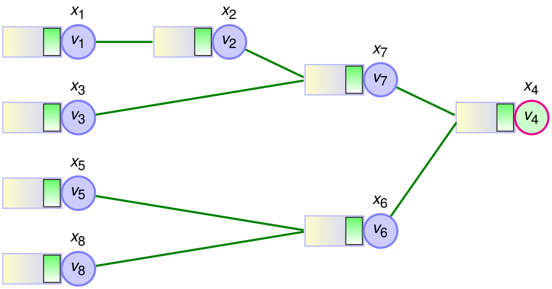

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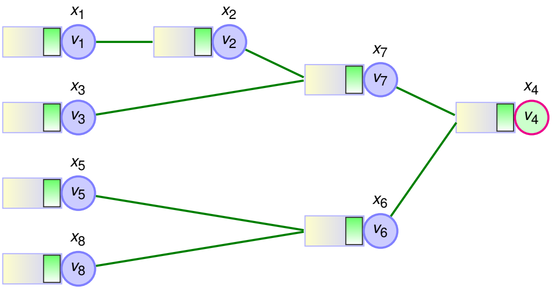

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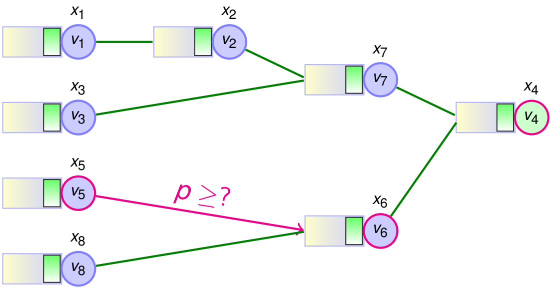

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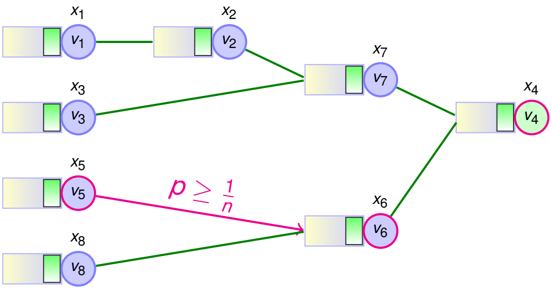

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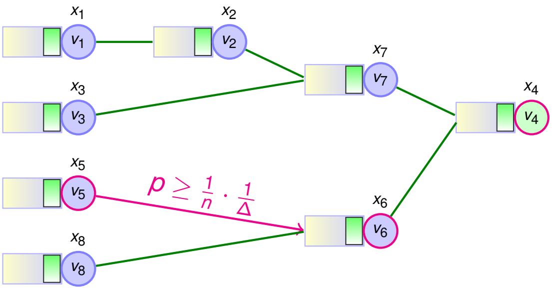

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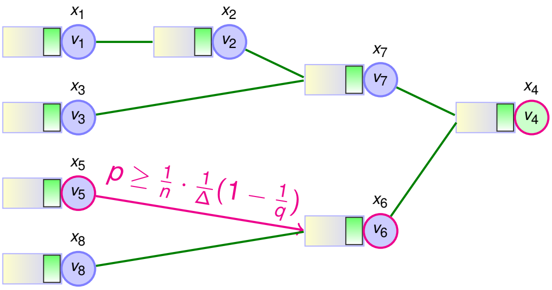

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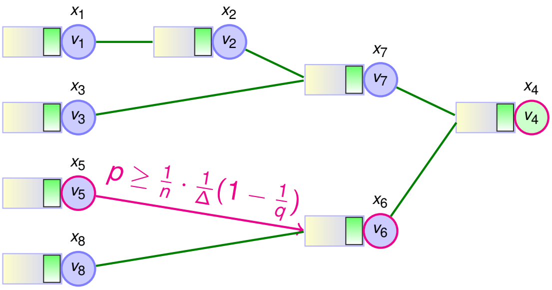

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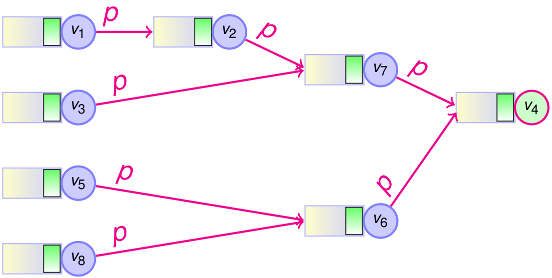
service time is geometrically distributed with p

Proof Overview – Exponential Servers Instead of Geometric



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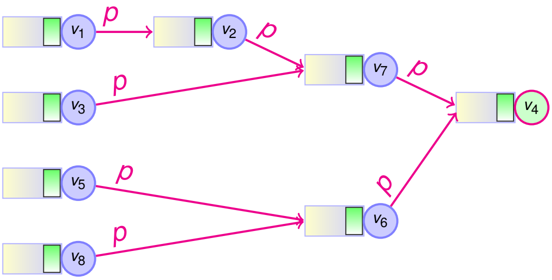

If $X \sim \text{Geom}(p)$, and $Y \sim \text{Exp}(p)$, then: $\Pr(Y > t) \geq \Pr(X > t)$

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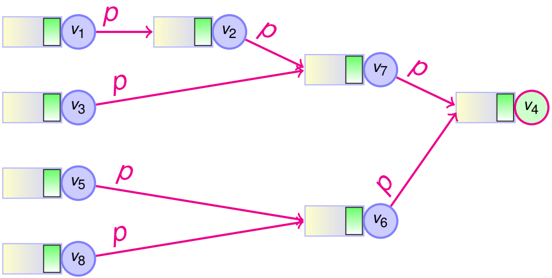

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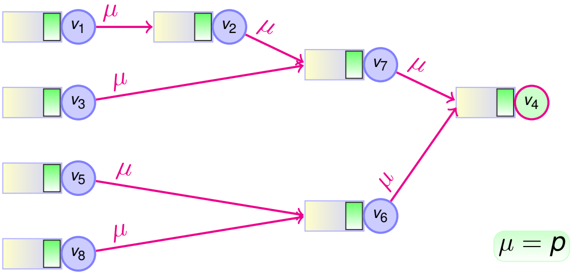

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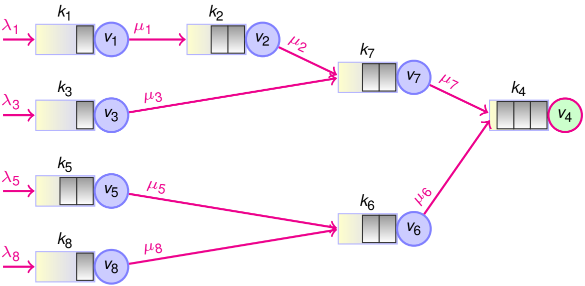
we replace servers, thus increasing the stopping time

Proof Overview – Jackson's Theorem

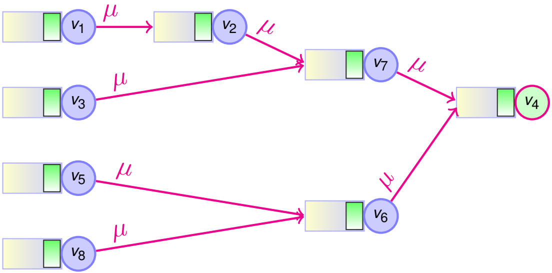


- **Jackson's Theorem:** If utilization at every queue is less than 1, the equilibrium state distribution of number of customers in each queue exists and it is given by:

For state $(k_1, k_2, \ldots, k_n)$, and utilization $\rho_i = \frac{\lambda_i}{\mu_i}$:

$$\pi(k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} \rho_i^{k_i}(1 - \rho_i).$$

Proof Overview – Applying Jackson's Theorem



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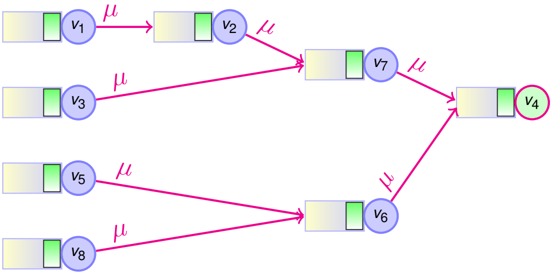

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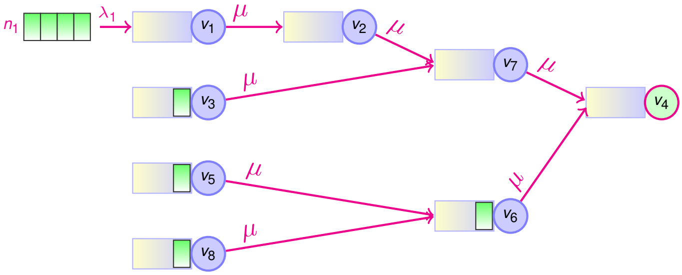

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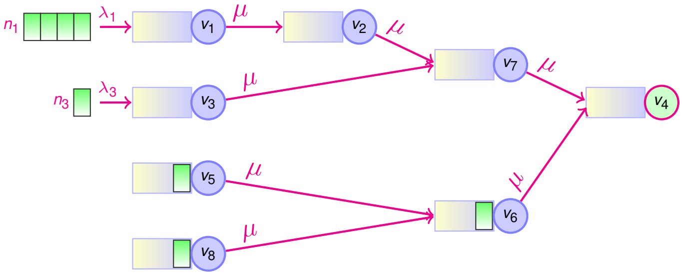

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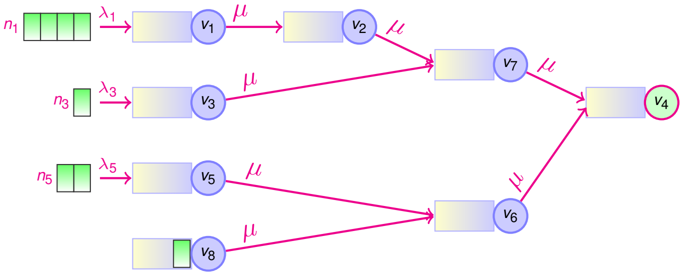

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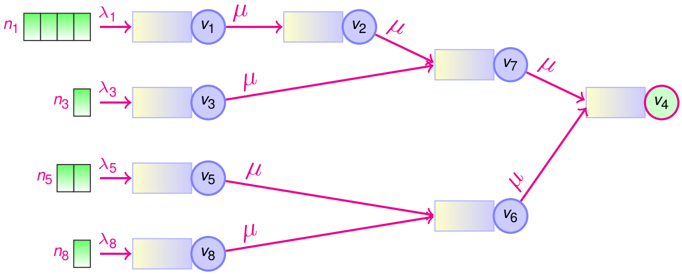

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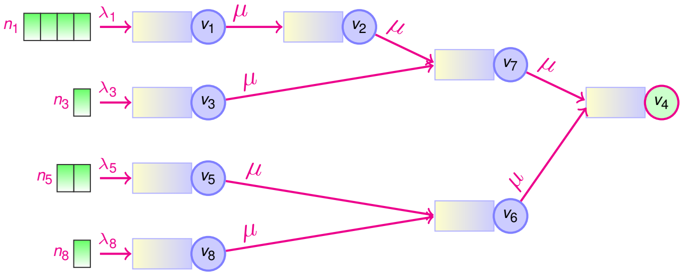

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by setting: $\lambda_i = \frac{\mu n_i}{2n}$, we obtain: $\rho_i < 1$

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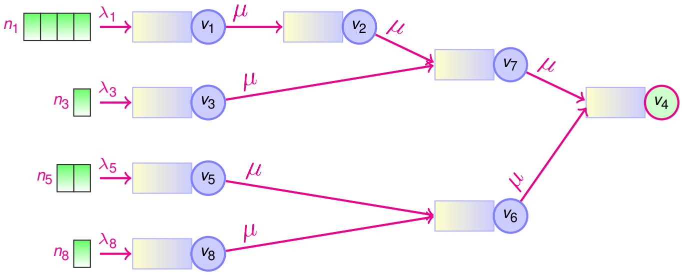

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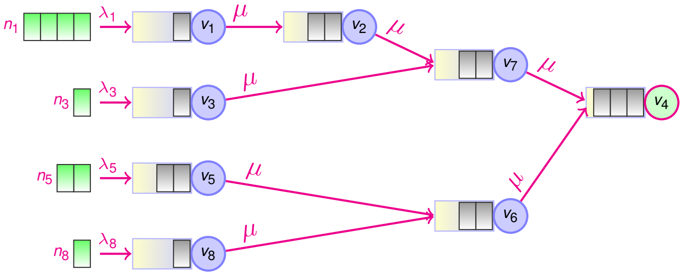

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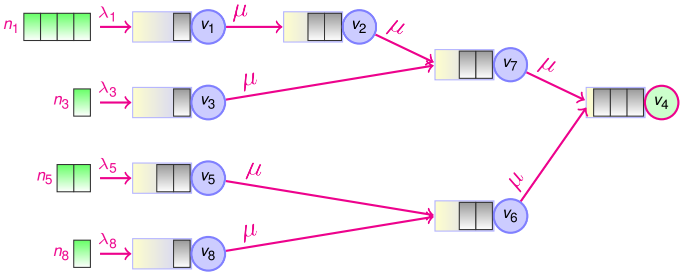

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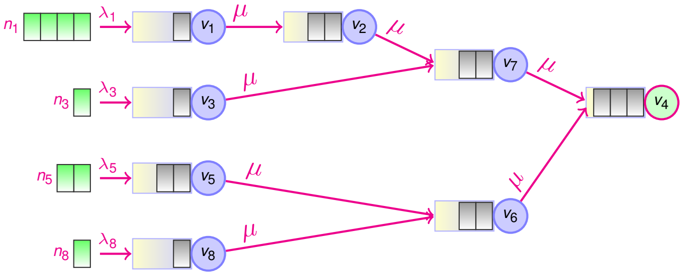

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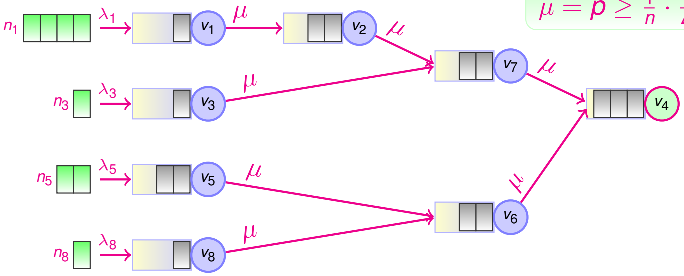
$\mu = p \geq \frac{1}{n} \cdot \frac{1}{\Delta}(1 - \frac{1}{q})$



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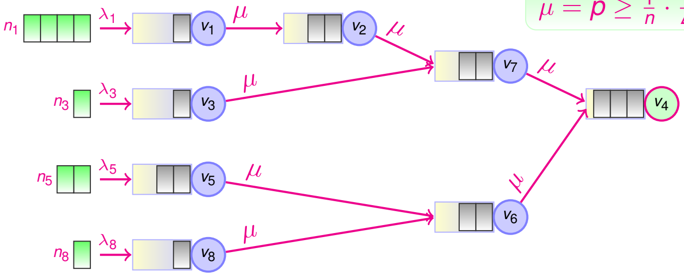

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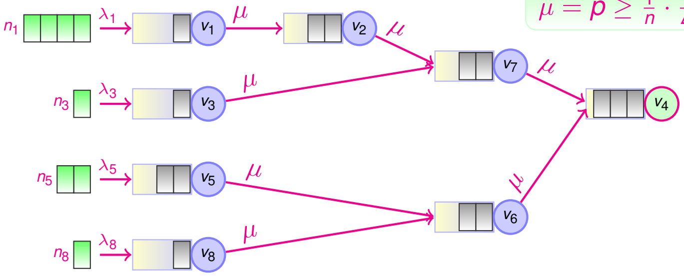


v_4 finishes after $O(\Delta n)$ rounds with exponential high probability

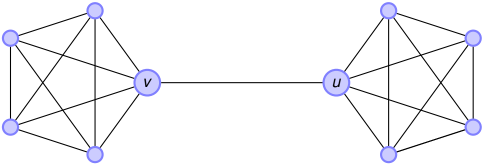
Using a union bound we obtain that the stopping time of all nodes is: $O(\Delta n)$ rounds

Proof Overview – Tightness of the Bound

Corollary 3

There exists a graph with n nodes, for which stopping time of algebraic gossip is $\Omega(n^2)$.

Proof Overview – Worst Case Lower Bound



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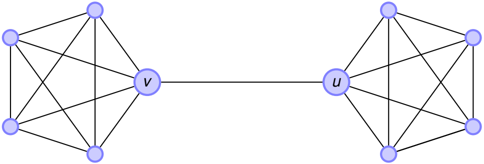
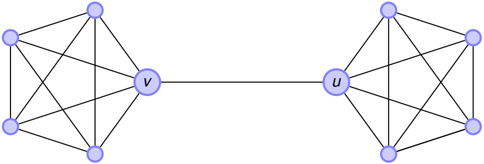


- Consider information flow form v to u.

Proof Overview – Worst Case Lower Bound



- Consider information flow form v to u.
- Only after receiving $\frac{n}{2}$ helpful messages, u finishes.

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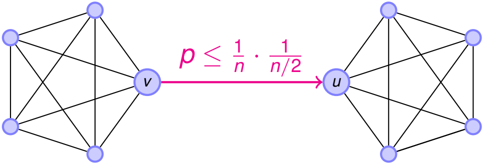

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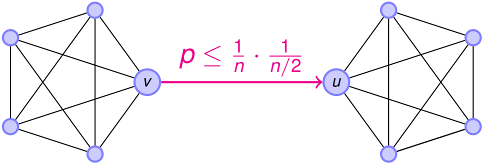

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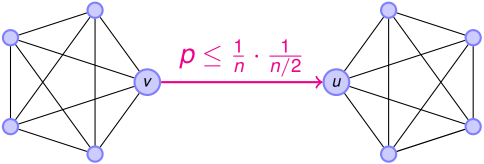

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- Using a Chernoff bound we obtain that u will finish after $\Omega(n^2)$ rounds.

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- Thank you!

Deb, S., Médard, M., and Choute, C. (2006).
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IEEE Transactions on Information Theory, 52(6):2486–2507.

Mosk-Aoyama, D. and Shah, D. (2006).
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